Math 206B Lecture 10 Notes

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1 Geometric RSK

1.1 Longest value of a path in RSK

Let Φ denote the R-S algorithm, and let $\hat{\Phi}$ denote RSK. For Φ, we had a property about the longest increasing subsequence of a permutation. What about for $\hat{\Phi}$?

Example 1.1. When we run RSK on

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 1 \\ 5 & 3 & 1 \end{bmatrix}$$

we get the pair of tableau

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 & 2 & 2 & 2 & 3 \\ \cline{1-5} 3 & 3 & 3 \\ \cline{1-3} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 3 & 3 \\ \cline{1-5} 3 & 3 & 3 \\ \cline{1-3} \end{array}$$

Then $\lambda_1 = 12$, which happens to be the sum of the numbers in the longest path from top left to bottom right : $2 + 1 + 5 + 3 + 1 = 12$.

Theorem 1.1. *Let $\hat{\Phi}(M) = (A, B)$ have shape λ. Then $\lambda = \gamma(M)$, where γ is the maximum total value of a path from $(1, 1)$ to (n, n).*

1.2 Geometric RSK

How would we feed a semistandard Young tableau to a computer? We want to think of $A = (\lambda \supseteq \cdots \supseteq \mu^{(2)} \supseteq \mu^{(1)})$, where μ_i is the shape of the tableau, only looking at the numbers $\leq i$.

Example 1.2. For the tableaux

1	1	1	1	1	1	1	1	2	2	2	3
2	2	2	2	3							
3	3	3									

1	1	1	1	1	2	2	2	3	3	3	3
2	2	2	3	3							
3	3	3									

we get

$$\begin{matrix} 12 & 11 & 8 \\ & 5 & 4 \\ & & 3 \end{matrix} \qquad \begin{matrix} 12 & & \\ 8 & 5 & \\ 5 & 3 & 3 \end{matrix}$$

We can put these together into a matrix

$$\begin{bmatrix} 12 & 11 & 8 \\ 8 & 5 & 4 \\ 5 & 3 & 3 \end{bmatrix}.$$

The output of RSK can then be sent to a computer as $x = (x_{i,j})$ such that

- $x_{i,j} \geq 0$
- $x_{i,j} \leq x_{i,j+1}, x_{i+1,j}$
- $\sum_{i-j=n-c} x_{i,j} = a_1 + \cdots + a_c$
- $\sum_{j-i=n-c} x_{i,j} = b_1 + \cdots + b_c$, where $0 \leq c \leq n$.

This defines a polytope. So we can think of $\tilde{\Phi} : \tilde{M}(\overline{a}, \overline{b}) \to \tilde{X}(a, b)$, where the left hand side takes a polytope defined by a matrix, and the the right side outputs a polytope defined by a matrix.

Theorem 1.2. *$\tilde{\Phi}$ is piecewise linear, volume preserving, and continuous.*

These polytopes were invented by Gelfand and Tseitlin. This was further developed in a paper by Gelfand and Zelevinsky.

1.3 Further generalization of RSK

Let $|\nu| = k$ be a Young diagram. Let

$$P_\nu(\overline{a}, \overline{b}) = \left\{ f : \nu \to \mathbb{R}_+ \text{ s.t. } f(i,j) \le f(i,j+1), f(i+1,j), \sum_j f(i,j) = a_i, \sum_i f(i,j) = b_j \right\},$$

$$Q_\nu(\overline{a}, \overline{b}) = \left\{ g : \nu \to \mathbb{R}_+ \text{ s.t. } \sum_{i-j=c} g(i,j) = d_c \,\forall c \right\},$$

where d_c is something.

Theorem 1.3. *There exists Φ^* sending $P_\nu(\overline{a}, \overline{b}) \to Q_\nu(\overline{a}, \overline{b})$ which is piecewise linear, continuous, volume preserving and a bijection $\Phi^* : P_\nu \cap \mathbb{Z}^k \to Q_\nu \cap \mathbb{Z}^k$.*